Parameterisation of observers for time delay systems and its application in observer design

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Abstract: The paper addresses the design of observers for systems with time delay. A factorisation approach is used to parameterise all observers for such systems. The necessary and sufficient condition of existence for observers is obtained. Based on the results in the parameterisation of observers, the estimation error dynamics is also parametrised. A design example is given to illustrate the proposed parameterisation procedure.

1 Introduction

This paper is concerned with the observer design for time delay systems. The state observation of time delay systems arises from the practical needs in system monitoring, regulation and/or identifying failure. In these cases, it is often desired to reconstruct the state variables of systems with time delay. In general, designing observers for reconstructing the state variables is more involved in a time delay system than it is in a delay-free system. To solve this problem, considerable efforts have been made during the last two decades [1–7], by using spectral decomposition [1–4], matrix fractional representation [5], and finite spectrum assignment [6, 7].

It is known that the parameterisation of all observers plays an important role in designing an observer, especially in designing an optimal robust observer under unknown disturbances [8–10]. However such parameterisation has not yet been developed for time delay systems. In this paper the factorisation approach is utilised to parameterise the set of *all* observers for the time delay systems.

It has been shown that the factorisation approach is a powerful tool in solving a variety of control system design problems [11]. Although the transfer function matrix of any finite-dimension system admits a proper stable Bezout factorisation, such factorisation does not, in general, exist for infinite-dimensional systems [12]. Thus, the proper stable Bezout factorisations of transfer function matrices have been studied for linear-invariant systems with commensurate time delays [12–14]. It is shown [13] that the existence of the proper stable Bezout factorisations is equivalent to the spectral controllability (or spectral observability) of the co-canonical (or canonical) realisation of a transfer function matrix. An explicit procedure for computing proper stable Bezout factorisations has already been given [13]. Recently, these results have been extended [14] using the finite-spectral assignment method [7, 15]. In this paper, some of their results are used to obtain the parameterisation of observers.

When disturbances exist in the time delay system, estimation errors could be caused. With the aid of the observer parameterisation proposed in this study, the parameterisation of estimation error dynamics with respect to the disturbances has also been obtained.

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2 Notation and preliminaries

Let **R** denote the field of real numbers, **C** denote the field of complex numbers or the complex plane, $\mathbf{C}_+$ ($\mathbf{C}_-$) denote the open right-half (left-half) plane, $\bar{\mathbf{C}}_+$ denote the closed right-half plane, and Ω denote either **C** or $\bar{\mathbf{C}}_+$. $\mathbf{R}[s]$ denote the set consisting of all finite sums $\Sigma_k a_k s^k$, where $a_k \in \mathbf{R}$ for all $k = 1, 2, ...$ Let z denote $\exp(-ds)$ where d is a fixed nonzero positive number and s is a complex variable. We also let $M(\cdot)$ denote the set of matrices, I denote the unity matrix, and 0 the null matrix.

Let $\mathbf{R}[z]$ denote the ring of polynomials in z with coefficients in **R**, and $\mathbf{R}(z)$ denote the field of rational functions in z with coefficients in **R**. Let $\mathbf{R}[z][s]$ denote the ring of polynomials in s with coefficients in $\mathbf{R}[z]$. Let $a(s, z)$ be an arbitrary element of $\mathbf{R}[z][s]$ given by $a(s, z) = a_n(z)s^n + a_{n-1}(z)s^{n-1} + \cdots + a_0(z)$, where $a_k(z) \in \mathbf{R}[z]$ for all k. The degree of $a(s, z)$ is the largest value of k for which $a_k(z) \neq 0$. The polynomial $a(s, z)$ is said to be *monic* in s if $a_n(z) = 1$, where n is the degree of $a(s, z)$. Let $\mathbf{R}[z](s)$ ($\mathbf{C}[z](s)$) denote the ring of rational functions in s with coefficients in $\mathbf{R}[z]$ ($\mathbf{C}[z]$).

Let $\mathbf{R}[s, z]$ denote the ring of polynomials in s and z with coefficients in **R**, and $\mathbf{R}(s, z)$ denote the field of rational functions in s and z with coefficients in **R**.

Let **P** denote the ring of rational functions defined by

$$\mathbf{P} = \{g(s,z) = \frac{b(s,z)}{a(s,z)} | a(s,z), b(s,z) \in \mathbf{R}[z][s]$$
$$a(s,z) \text{ is monic in } s, \text{ and}$$
$$g(s,z) \text{ is strictly proper in } s\}$$

Then $G(s, z) \in M(\mathbf{P})$ can be viewed as the transfer function matrix of an m-input and p-output linear time-invariant time delay system. A triple ($C(z)$, $A(z)$, $B(z)$) of matrices over $\mathbf{R}[z]$ is a realisation of $G(s, z)$ if and only if $G(s, z) = (C(z)(sI - A(z))^{-1}B(z)$.

2.1 Definition 1 [12, 14]

(i) The pair ($A(z)$, $B(z)$) is $\mathbf{R}[z]$-controllable if and only if $rank[sI - A(z)\ B(z)] = n$ for all $(s, z) \in \mathbf{C}^2$.

(ii) The pair ($A(z)$, $B(z)$) is $\mathbf{R}(z)$-controllable if and only if $rank[sI - A(z)\ B(z)] = n$ for all but finite many pair $(s, z) \in \mathbf{C}^2$.

(iii) The pair ($A(z)$, $B(z)$) is spectrally controllable over Ω if and only if $rank[sI - A(\exp(-sd)\ B(\exp(-sd))] = n$ for all $s \in \mathbf{C}$.

(iv) The pair ($C(z)$, $A(z)$) is $\mathbf{R}[z]$-observable if and only if ($A'(z)$, $C'(z)$) is $\mathbf{R}[z]$-controllable.

(v) The pair ($C(z)$, $A(z)$) is $\mathbf{R}(z)$-observable if and only if ($A'(z)$, $C'(z)$) is $\mathbf{R}[z]$-controllable.

(vi) The pair ($C(z)$, $A(z)$) is spectrally observable over Ω if and only if ($A'(z)$, $C'(z)$) is spectrally controllable over Ω.

2.2 Definition 2 [12–14]

(i) The triple ($C(z)$, $A(z)$, $B(z)$) is canonical if and only if ($A(z)$, $B(z)$) is $\mathbf{R}[z]$-controllable and ($C(z)$, $A(z)$) is $\mathbf{R}(z)$-observable.

(ii) The triple ($C(z)$, $A(z)$, $B(z)$) is co-canonical if and only if ($B'(z)$, $A'(z)$, $C'(z)$) is canonical.

(iii) The triple ($C(z)$, $A(z)$, $B(s)$) is spectrally canonical over Ω if and only if ($A(z)$, $B(z)$) is spectrally controllable over Ω and ($C(z)$, $A(z)$) is spectrally observable over Ω.

It is known that any transfer function matrix $G(s, z)$ has both a canonical realisation and a co-canonical realisation. Therefore the spectrally canonical realisations can be studied in term of the canonical or co-canonical realisation of $G(s, z)$ and the next result is known.

2.3 Lemma 1 [13]

Suppose $G(s, z) \in M(\mathbf{P})$ then the following three statements are equivalent:

(i) $G(s, z)$ has a spectrally canonical realisation over Ω.

(ii) Any canonical realisation of $G(s, z)$ is spectrally observable over Ω.

(iii) Any co-canonical realisation of $G(s, z)$ is spectrally controllable over Ω.

The following definitions are related with the coefficient matrices of the Bezout identity to be defined over a ring which contains both pure and distributed time delays.

Let $\Theta(z)$ denote the ring of polynomials in z, and $\Theta[z][s]$ denote the set consisting of all finite sums $\Sigma a_i(s, z)s^i$ where $a_i(s, z) \in \Theta[z]$ for all $i (i = 1, 2, ...)$. Any element $a(s, z) \in \Theta[z][s]$ has unique representation as polynomials in s [13]. A monic polynomial $a(s, z) \in \Theta[z][s]$ is stable if and only if $a(s, \exp(-sd)) \neq 0$ for all $s \in \bar{\mathbf{C}}_+$. It is known that in general $a(s, \exp(-sd))$ has infinitely many zeros. In particular, $a(s, z)$ is finite-spectrally stable if and only if $a(s, z)$ is stable and $a(s, \exp(-sd))$ has only finitely many zeros.

Let $\mathbf{S}_{IN}$ and $\mathbf{S}_{FI}$ denote the ring of proper stable functions defined by:

$$\mathbf{S}_{IN} = \{q(s,z) = \frac{g(s,z)}{f(s,z)} | f(s,z), g(s,z) \in \Theta[z][s]$$
$$f(s,z) \text{ is monic in } s \text{ and stable, and}$$
$$q(s,z) \text{ is proper in } s\} \tag{1}$$

$$\mathbf{S}_{FI} = \{q(s,z) = \frac{g(s,z)}{f(s)} | g(s,z) \in \Theta[z][s], f(s) \in \mathbf{R}[s]$$
$$f(s) \text{ is monic in } s \text{ and stable, and}$$
$$q(s,z) \text{ is proper in } s\} \tag{2}$$

respectively. Note that the element of $\mathbf{S}_{IN}$ has possibly infinitely many poles, and every element $\mathbf{S}_{FI}$ has only finitely many poles.

It has been shown [14] that any $G(s, z) \in M(\mathbf{P})$ has both a Bezout factorisation in $\mathbf{S}_{IN}$ and in $\mathbf{S}_{FI}$ if $G(s, z)$ is spectrally canonical over Ω.

2.4 Lemma 2 [14]

Suppose $G(s, z) \in M(\mathbf{P})$ is spectrally canonical over Ω. A double co-prime factorisation of $G(s, z)$ can be written as:

$$G(s,z) = N(s,z)M^{-1}(s,z) = \tilde{M}^{-1}(s,z)\tilde{N}(s,z) \tag{3}$$

where $N(s, z)$ $M(s, z)$ and $\tilde{N}(s, z)$, $\tilde{M}(s, z)$ are right and left coprime $M(\mathbf{S}_{IN})$ matrices, respectively. For this double coprime factorisation, there exist $M(\mathbf{S}_{IN})$ matrices $Y(s, z)$, $X(s, z)$ and $\tilde{Y}(s, z)$, $\tilde{X}(s, z)$ satisfying:

$$\begin{bmatrix} Y(s,z) & X(s,z) \\ -\tilde{N}(s,z) & \tilde{M}(s,z) \end{bmatrix} \begin{bmatrix} M(s,z) & -\tilde{X}(s,z) \\ N(s,z) & \tilde{Y}(s,z) \end{bmatrix} = I \tag{4}$$

which is referred to as doubly Bezout factorisation (DBF) in $\mathbf{S}_{IN}$.

The state-space realisation of the eight factors in eqn. 4 can be calculated as [14]:

$$\begin{aligned} M(s,z) &= F_e(s,z)\big(sI_e - A_0(s,z)\big)^{-1}B_e(z) + I \\ N(s,z) &= C_e(z)\big(sI_e - A_0(s,z)\big)^{-1}B_e(z) \end{aligned} \tag{5}$$

$$\begin{aligned} \tilde{M}(s,z) &= C_e(z)\big(sI_e - \tilde{A}_0(z)\big)^{-1}K_e(s,z) + I \\ \tilde{N}(s,z) &= C_e(z)\big(sI_e - \tilde{A}_0(z)\big)^{-1}B_e(z) \end{aligned} \tag{6}$$

$$\begin{aligned} Y(s,z) &= -F_e(s,z)\big(sI_e - \tilde{A}_0(z)\big)^{-1}B_e(z) + I \\ X(s,z) &= F_e(s,z)\big(sI_e - \tilde{A}_0(z)\big)^{-1}K_e(z) \end{aligned} \tag{7}$$

$$\begin{aligned} \tilde{Y}(s,z) &= -C_e(z)\big(sI_e - A_0(s,z)\big)^{-1}K_e(z) + I \\ \tilde{X}(s,z) &= F_e(s,z)\big(sI_e - A_0(s,z)\big)^{-1}K_e(z) \end{aligned} \tag{8}$$

where

$$A_0(s,z) = A_e(z) + B_e(z)F_e(s,z) \tag{9}$$

$$\tilde{A}_0(z) = A_e(z) + K_e(z)C_e(z) \tag{10}$$

and

$$\begin{aligned} A_e(s,z) &= \begin{bmatrix} A(z) & B(z) \\ 0 & -I \end{bmatrix} B_e(z) = \begin{bmatrix} 0 \\ I \end{bmatrix} \\ C_e(z) &= \begin{bmatrix} C(z) & 0 \end{bmatrix} \end{aligned} \tag{11}$$

$$\begin{aligned} F_e(s,z) &= [\, F_1(s,z) \quad F_2(s,z) \,] \quad K_e(z) = \begin{bmatrix} K_2(z) \\ 0 \end{bmatrix} \\ I_e &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{12}$$

Given $\alpha_i[z] \in \mathbf{R}[z]$ and $\beta_i[z] \in \mathbf{R}[z]$ $(i = 1, 2, ..., n)$, then $F_e(s, z) \in M(\Theta(z))$ and $K_e(z) \in M(\mathbf{R}[z])$ can be chosen such that:

$$\det\big(sI_e - A_0(s,z)\big) = \prod_{i=1}^{n}(s + \alpha_i(z)) \tag{13}$$

$$\det\left(sI_e - \tilde{A}_0(z)\right) = \prod_{i=1}^{n}(s+\beta_i(z)) \quad (14)$$

are stable. Then from eqn. 9, we have

$$G(s,z) = C_e(z)\left(sI_e - A_e(z)\right)^{-1}B_e(z) \quad (15)$$

When choosing positive real numbers $\alpha_i[z] \in \mathbf{R}$ and $\beta_i[z] \in \mathbf{R}$ (i = 1, 2, ..., n), $\mathbf{S}_{IN}$ can be replaced by $\mathbf{S}_{FI}$ in Lemma 2.

2.5 *Lemma 3 [14]*

Suppose $G(s, z) \in M(\mathbf{P})$ is spectrally canonical over Ω. $G(s, z)$ has a DBF in $\mathbf{S}_{FI}$.

In the proceeding Sections, only $\mathbf{S}_{IN}$ will be addressed. However, all the results also hold if $\mathbf{S}_{IN}$ is substituted for $\mathbf{S}_{FI}$.

3 Problem formulation

Consider the following linear time delay system described by:

$$\dot{x}(t) = A(z)x(t) + B(z)u(t) \quad (16)$$

$$y(t) = C(z)x(t) \quad (17)$$

$$r(t) = E(z)x(t) \quad (18)$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the input vector, $y(t) \in \mathbf{R}^P$ is the measured output vector, $r(t) \in \mathbf{R}^k$ is the state to be estimated, and $A(z)$, $B(z)$, $C(z)$, $E(z) \in \mathbf{R}[z]$ are constant matrices with appropriate dimensions, and z denotes a formal delay operator which has the property such that $zx(t) = x(t-d)$, $z^2x(t) = x(t-2d)$, ..., for delay duration d. Taking Laplace transforms of eqns. 16–18, the transfer function matrix description is given by:

$$y(s) = G(s,z)u(s) \quad (19)$$

$$r(s) = E(z)x(s) \quad (20)$$

with

$$G(s,z) = C(z)(sI - A(z))^{-1}B(z) \quad (21)$$

and note that $z = \exp(-sd)$ in the Laplace domain.

The observer for eqns. 16–18 can be described by

$$\hat{r}(s) = F(s,z)u(s) + H(s,z)y(s) \quad (22)$$

where $F(s, z) \in M(\mathbf{S}_{IN})$ and $H(s, z) \in M(\mathbf{S}_{IN})$. The estimation error for $r(t)$ using the observer eqn. 22 should satisfy

$$\lim_{t\to\infty}(r(t)) - \hat{r}(t)) = 0 \quad (23)$$

for all $u(t)$ and initial states.

Suppose $G(s, z) \in M(\mathbf{P})$ is spectrally canonical over Ω. A right co-prime factorisation of $G(s, z)$ can be written as

$$G(s,z) = N(s,z)M^{-1}(s,z) \quad (24)$$

where $N(s, z)$ and $M(s, z)$ is $\mathbf{S}_{IN}$ matrices.

Introducing the partial state $\xi(s)$, we can rewrite eqn. 19 with the factorisation eqn. 24 as

$$M(s,z)\xi(s) = u(s) \quad (25)$$

$$N(s,z)\xi(s) = y(s) \quad (26)$$

Correspondingly, the variable $r(s) = E(z)x(s)$ in eqn. 20 can be expressed as

$$r(s) = P(s,z)\xi(s) \quad (27)$$

with

$$P(s,z) = E_e(z)\left(sI_e - A_0(s,z)\right)^{-1}B_e(z) \quad (28)$$

where $E_e(z) = E(z)$ 0]. Eqn. 28 can be proved as follows.

From eqns. 21 and 24, we obtain

$$C(z)(sI - A(z))^{-1}B(z) = N(s,z)M^{-1}(s,z) \quad (29)$$

Using eqn. 5 we have

$$\left(sI - A(z)\right)^{-1}B(z) = (I \quad 0)\left(sI_e - A_0(s,z)\right)^{-1}B_e(z)M^{-1}(s,z) \quad (30)$$

which leads to

$$\begin{aligned} x(s) &= (sI - A(z))^{-1}B(z)u(s) \\ &= (I \quad 0)\left(sI_e - A_0(s,z)\right)^{-1}B_e(z)M^{-1}(s,z)u(s) \\ &= (I \quad 0)\left(sI_e - A_0(s,z)\right)^{-1}B_e(z)\xi(s) \end{aligned} \quad (31)$$

Thus the variable $E(z)x(s)$ can be expressed by

$$\begin{aligned} r(s) &= E(z)x(s) = (E(z) \quad 0)\left(sI_e - A_0(s,z)\right)^{-1}B_e(z)\xi(s) \\ &= E_e(z)\left(sI_e - A_0(s,z)\right)^{-1}B_e(z)\xi(s) \end{aligned} \quad (32)$$

For this new system description eqns. 25–27, we can obtain the following existence condition of observers.

3.1 *Theorem 1*

For the given system eqns. 16–18, the variable $r(t) = E(z)x(t)$ can be observed using the observer eqn. 7 if and only if the following condition holds:

$$F(s,z)M(s,z) + H(s,z)N(s,z) = P(s,z) \quad (33)$$

3.1.1 Proof: Necessity. The estimation error in eqn. 23 is given as

$$r(t) - \hat{r}(t) = L^{-1}[r(s) - \hat{r}(s)] \quad (34)$$

where $L^{-1}[\cdot]$ denotes inverse Laplace transformation. Furthermore, let the state-space realisation of $F(s, z)$ and $H(s, z)$ in the observer eqn. 22 be ($A_f(z)$, $B_f(z)$, $C_f(z)$, $D_f(z)$) and ($A_h(z)$, $B_h(z)$ $C_h(z)$ $D_h(z)$) with stable vectors x_f and x_h and initial values $x_f(0)$ and $x_h(0)$, respectively. Then we have:

$$L^{-1}[r(s)] = L^{-1}[P(s,z)\xi(s) + E_e(z)\left(sI - A_0(z)\right)^{-1}x(0)] \quad (35)$$

$$\begin{aligned} L^{-1}[\hat{r}(s)] &= L^{-1}[F(s,z)u(s) + C_f\left(sI - A_f(z)\right)^{-1}x_f(0) \\ &\qquad + H(s,z)y(s) + C_h\left(sI - A_h(z)\right)^{-1}x_h(0)] \\ &= L^{-1}[\left(F(s,z)M(s,z) + H(s,z)N(s,z)\xi(s)\right. \\ &\qquad + C_f\left(sI - A_f(z)\right)^{-1}x_f(0) + C_h\left(sI - A_h(z)\right)^{-1}x_h(0)] \end{aligned} \quad (36)$$

Substituting eqns. 35 and 36 into eqn. 34, gives:

$$\begin{aligned} r(t) - \hat{r}(t) = L^{-1}[(P(s,z) &- F(s,z)M(s,z) \\ &- H(s,z)N(s,z))\xi(s) + E(sI - A_0(z))^{-1}x(0) \\ &- C_f(sI - A_f(z))^{-1}x_f(0) \\ &- C_h(sI - A_h(z))^{-1}x_h(0)] \end{aligned}$$

Since $E(sI - A_0(z))^{-1}$, $C_f(sI - A_f(z))^{-1}$ and $C_h(sI - A_h(z))^{-1} \in M(\mathbf{S}_{IN})$, we have:

$$\begin{aligned} &\lim_{t\to\infty} L^{-1}[E\left(sI - A_0(z)\right)^{-1}x(0) \\ &- C_f\left(sI - A_f(z)\right)^{-1}x_f(0) - C_h\left(sI - A_h(z)\right)^{-1}x_h(0)] = 0 \end{aligned}$$

Therefore, the condition eqn. 33 is a necessary condition for system eqn. 22 to satisfy condition eqn. 23.

3.1.2 Sufficiency:
It is known that the observer satisfies

$$[F(s,z)\ H(s,z)]\begin{bmatrix} M(s,z) \\ N(s,z) \end{bmatrix} = P(s,z) \quad (37)$$

That is

$$[F(s,z)\ H(s,z)]\begin{bmatrix} M(s,z) \\ N(s,z) \end{bmatrix}\xi(s) = P(s,z)\xi(s)$$

then

$$[F(s,z)\ H(s,z)]\begin{bmatrix} u(s) \\ y(s) \end{bmatrix} = P(s,z)\xi(s) \quad (38)$$

It is seen that the term $P(s, z)\xi(s)$ on the right side of eqn. 38 is the dynamics of the state function $E(z)x(t)$ in the frequency domain. Therefore, the observer can be expressed as the form of eqn. 22. Thus the proof is completed.

The objective in the next Section is to find the set of *all* observers which satisfies condition eqn. 33 (e.g. to parameterise all observers). The factorisation approach for time delay systems will be used to solve this problem.

4 Parameterisation of observers

4.1 Theorem 2
Given plant eqns. 16–18. Suppose $G(s, z) \in M(\mathbf{P})$ is spectrally canonical over Ω, with the right and left co-prime factorisation $(N(s, z), M(s, z))$, $(\tilde{N}(s, z), \tilde{M}(s, z))$ in $\mathbf{S}_{IN}$, respectively. Then the set of all observers for $z(t)$ is parameterised by:

$$F(s,z) = \left[P(s,z)Y(s,z) - Q(s,z)\tilde{N}(s,z)\right] \quad (39)$$

$$H(s,z) = \left[P(s,z)X(s,z) + Q(s,z)\tilde{M}(s,z)\right] \quad (40)$$

$$Q(s,z) \in M(\mathbf{S}_{IN}) \quad (41)$$

where $Y(s, z)$, $X(s, z)$ satisfy the Bezout identity eqn. 4 corresponding the co-prime factorisation of $G(s, z)$.

4.1.1 Proof:
Necessity. Select a $Q(s, z)$ satisfying eqn. 41. There exists an observer,

$$\begin{aligned} r(s) &= F(s,z)u(s) + H(s,z)y(s) \\ &= \left[P(s,z)Y(s,z) - Q(s,z)\tilde{N}(s,z)\right]u(s) \\ &\quad + \left[P(s,z)X(s,z) + Q(s,z)\tilde{M}(s,z)\right]y(s) \quad (42) \end{aligned}$$

or

$$\begin{aligned} &[F(s,z)\ H(s,z)]\left[P(s,z)Y(s,z) - Q(s,z)\tilde{N}(s,z)\right. \\ &\qquad\qquad \left.P(s,z)X(s,z) + Q(s,z)\tilde{M}(s,z)\right] \quad (43) \\ &= [P(s,z)\ Q(s,z)]\begin{bmatrix} Y(s,z) & X(s,z) \\ -\tilde{N}(s,z) & \tilde{M}(s,z) \end{bmatrix} \end{aligned}$$

Thus

$$\begin{aligned} &[F(s,z)\ H(s,z)]\begin{bmatrix} M(s,z) \\ N(s,z) \end{bmatrix} \\ &= [P(s,z)\ Q(s,z)]\begin{bmatrix} Y(s,z) & X(s,z) \\ -\tilde{N}(s,z) & \tilde{M}(s,z) \end{bmatrix}\begin{bmatrix} M(s,z) \\ N(s,z) \end{bmatrix} \\ &= [P(s,z)\ Q(s,z)]\begin{bmatrix} I \\ 0 \end{bmatrix} = P(s,z) \end{aligned} \quad (44)$$

It can be seen that this satisfies the condition eqn. 33 of the observation.

4.1.2 Sufficiency:
The observer is given by:

$$r(s) = F(s,z)u(s) + H(s,z)y(s)$$

It is desired to find a $Q(s, z) \in M(\mathbf{S}_{IN})$ such that $F(s, z)$ and $H(s, z)$ can be expressed as eqns. 39 and 40, that is:

$$[P(s,z)\ Q(s,z)]\begin{bmatrix} Y(s,z) & X(s,z) \\ -\tilde{N}(s,z) & \tilde{M}(s,z) \end{bmatrix} = [F(s,z)\ H(s,z)] \quad (45)$$

It is known that the observer satisfies:

$$[F(s,z)\ H(s,z)]\begin{bmatrix} M(s,z) \\ N(s,z) \end{bmatrix} = P(s,z)$$

From Lemma 2,

$$\begin{bmatrix} Y(s,z) & X(s,z) \\ -\tilde{N}(s,z) & \tilde{M}(s,z) \end{bmatrix}$$

is nonsingular. Let

$$\begin{bmatrix} Y(s,z) & X(s,z) \\ -\tilde{N}(s,z) & \tilde{M}(s,z) \end{bmatrix}^{-1} = \begin{bmatrix} M(s,z) & -\tilde{X}(s,z) \\ N(s,z) & \tilde{Y}(s,z) \end{bmatrix} \quad (46)$$

This is a stable matrix [12]. From eqn. 45, we obtain

$$\begin{aligned} [P(s)\ Q(s)] &= [F(s)\ H(s)]\begin{bmatrix} Y(s,z) & X(s,z) \\ -\tilde{N}(s,z) & \tilde{M}(s,z) \end{bmatrix}^{-1} \\ &= [F(s)\ H(s)]\begin{bmatrix} M(s,z) & -\tilde{X}(s,z) \\ N(s,z) & \tilde{Y}(s,z) \end{bmatrix} \end{aligned} \quad (47)$$

That is,

$$\begin{aligned} P(s,z) &= [F(s,z)\ H(s,z)]\begin{bmatrix} M(s,z) \\ N(s,z) \end{bmatrix} \\ Q(s,z) &= [F(s,z)\ H(s,z)]\begin{bmatrix} -\tilde{X}(s,z) \\ \tilde{Y}(s,z) \end{bmatrix} \end{aligned} \quad (48)$$

Thus eqn. 45 is satisfied.

Theorem 2 gives the result of parameterisation of all observers for the time delay system in eqns. 16–18. According to this parameterisation, the observer design reduces to searching for a suitable parameterisation matrix in $\mathbf{S}_{IN}$ set. This provides us with a systematic procedure to design observers. As the summary, the procedure designing observers is given in the following algorithm:

(i) Obtain a co-prime factorisation of $G(s, z) \in M(\mathbf{P})$ for $\tilde{N}(s, z)$ and $\tilde{M}(s, z)$.

(ii) Calculate $Y(s, z)$ and $X(s, z)$ which satisfy the Bezout identity eqn. 4 and $P(s, z)$ given in eqn. 28.

(iii) Choose the parameterisation matrix $Q(s, z) \in M(\mathbf{S}_{IN})$.

(iv) Observers are given in the following form

$$\begin{aligned} \hat{r}(s) &= \left[P(s,z)\,Y(s,z) - Q(s,z)\,\tilde{N}(s,z)\right]u(s) \\ &\quad + \left[P(s,z)\,X(s,z) + Q(s,z)\,\tilde{M}(s,z)\right]y(s) \end{aligned}$$

4.2 Remark 1
The above algorithm is based on the proper stable Bezout factorisation under state-space representation, and therefore, it can be easily realised and implemented.

4.3 Remark 2
The selection of the parameterisation matrix $Q(s, z)$ depends on the performance specification of observer design. When a certain design specification is given, one can solve the corresponding observer design by finding an optimal $Q(s, z)$.

5 Parameterisation of estimation error dynamics

When disturbances exist in the system, the performance in eqn. 23 will never be satisfied. In this case, we can give the parameterisation of estimation error dynamics as follows.

When there are disturbances in given plants, the description for the system in eqns. 16–18 becomes:

$$\dot{x}(t) = A(z)x(t) + B(z)u(t) + U(z)d(t) \tag{49}$$

$$y(t) = C(z)x(t) \tag{50}$$

$$r(t) = E(z)x(t) \tag{51}$$

where $d(t) \in \mathbf{R}^q$ is the unknown disturbance vector, $U(z) \in \mathbf{R}[z]$ are constant matrices with appropriate dimensions. Then the transfer function description is rewritten by

$$y(s) = G(s,z)u(s) + G_d(s,z)d(s) \tag{52}$$

where $G(s, z)$ is expressed in eqn. 21 and

$$G_d(s,z) = C(z)(sI - A(z))^{-1}U(z) \tag{53}$$

The double coprime factorisation of $G_d(s, z)$ can be written as

$$G_d(s,z) = N_d(s,z)M_d^{-1}(s,z) = \tilde{M}_d^{-1}(s,z)\tilde{N}_d(s,z) \tag{54}$$

According to eqn. 11, eqn. 53 also can be expressed as

$$G_d(s,z) = C_e(z)\big(sI_e - A_d(z)\big)^{-1}U_e(z) \tag{55}$$

where

$$A_d(z) = \begin{bmatrix} A(z) & U(z) \\ 0 & -I \end{bmatrix} \quad U_e(z) = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{56}$$

Furthermore, through introducing the partial state $\xi(s)$ defined in eqn. 25, we have

$$y(s) = N(s,z)\xi(s) + G_d(s,z)d(s) \tag{57}$$

and

$$r(s) = P(s,z)\xi(s) + F_d(s,z)d(s) \tag{58}$$

where $P(s, z)$ is given in eqn. 28 and

$$\begin{aligned} F_d(s,z) &= E(z)(sI - A(z))^{-1}U(z) \\ &= E_e(z)\big(sI_e - A_d(z)\big)^{-1}U_e(z) \end{aligned} \tag{59}$$

Eqn. 59 can be obtained as follows.

From eqns. 53 and 54, we obtain

$$sI - A(z))^{-1}U(z) = (I \quad 0)\,\big(sI_e - A_d(z)\big)^{-1}U_e(z) \tag{60}$$

Furthermore, we have

$$\begin{aligned} x(s) &= (sI - A(z))^{-1}B(z)u(s) + (sI - A(z))^{-1}U(s)d(s) \\ &= (I \quad 0)\,\big(sI_e - A_0(s,z)\big)^{-1}B_e(z)M^{-1}(s,z)u(s) \\ &\quad + (I \quad 0)\,\big(sI_e - A_d(z)\big)^{-1}U_e(z)d(s) \end{aligned} \tag{61}$$

Then

$$\begin{aligned} r(s) &= E_e(z)\big(sI_e - A_0(s,z)\big)^{-1}B_e(z)\xi(s) \\ &\quad + E_e(z)\big(sI_e - A_d(z)\big)U_e(z)d(s) \end{aligned} \tag{62}$$

Thus eqn. 59 is obtained.

The estimation error is described by

$$e(s) = r(s) - \hat{r}(s) \tag{63}$$

Substituting $F(s, z)$ and $H(s, z)$ of eqns. 39 and 40 into eqn. 63, we obtain

$$\begin{aligned} e(s) = \big((&F_d(s,z) - P(s,z)X(s,z)G_d(s,z) \\ &- Q(s,z)\tilde{M}(s,z)G_d(s,z)\big)d(s) \end{aligned}$$

That is,

$$e(s) = \big(T_1(s,z) - Q(s,z)T_2(s,z)\big)d(s) \tag{64}$$

where

$$\begin{aligned} T_1(s,z) &= F_d(s,z) - P(s,z)X(s,z)G_d(s,z) \\ T_2(s,z) &= \tilde{M}(s,z)G_d(s,z) \end{aligned} \tag{65}$$

The following theorem can be proved.

5.1 Theorem 3

Given system eqns. 49–51. The set of all achievable transfer function matrices of the estimation errors $e(t)$ in the presence of the disturbance vector $d(t)$ is parameterised as

$$T(s,z) = T_{11}(s,z) - R(s,z)T_{22}(s,z) \tag{66}$$

where:

$$\begin{aligned} T_{11}(s,z) &= E_e(z)\big(sI_e - \tilde{A}_0(z)\big)^{-1}\begin{pmatrix} U(z) - B(z) \\ I \end{pmatrix} \\ &\in M(\mathbf{S}_{IN}) \end{aligned} \tag{67}$$

$$\begin{aligned} T_{22}(s,z) &= C_e(z)\big(sI_e - \tilde{A}_0(z)\big)^{-1}\begin{pmatrix} U(z) - B(z) \\ I \end{pmatrix} \\ &\in M(\mathbf{S}_{IN}) \end{aligned} \tag{68}$$

$$\begin{aligned} R(s,z) &= E_e(z)\big(sI_e - A_0(s,z)\big)^{-1}K_e(z) + Q(s,z) \\ &\in M(\mathbf{S}_{IN}) \end{aligned} \tag{69}$$

5.1.1 Proof:

See the Appendix. Eqn. 66 in Theorem 3 gives a straightforward relationship between the estimation error and disturbance vector. $R(s, z)$ is the only unknown parameterisation matrix in eqn. 66. Thus, when a certain design specification is used one can solve the optimal robust observation problem by finding $R(s, z)$. It can be seen that the parameterisation of observer provides a basic tool for this kind of optimal problem.

5.2 Remark 3

Based on the parameterisation of observer, we can obtain a clearly observer construction. From eqns. 22, 39 and 40, we have

$$\begin{aligned} \hat{r}(s) &= F(s,z)u(s) + H(s,z)y(s) \\ &= \Big[P(s,z)Y(s,z) - Q(s,z)\tilde{N}(s,z)\Big]\,u(s) \\ &\quad + \Big[P(s,z)X(s,z) + Q(s,z)\tilde{M}(s,z)\Big]\,y(s) \end{aligned} \tag{70}$$

From eqn. 4, $Y(s, z) = M^{-1}(s, z) - X(s, z)N(s, z)M^{-1}(s, z)$ and substitute it into eqn. 70, gives

$$\begin{aligned} \hat{r}(s) &= P(s,z)M^{-1}(s,z)u(s) \\ &\quad + (P(s,z)X(s,z) + Q(s,z)\tilde{M}(s,z))\,[y(s) - G(s,z)u(s)] \end{aligned} \tag{71}$$

Thus, we have

$$\begin{aligned} \hat{r}(s) = P(s,z)&M^{-1}(s,z)u(s) + L(s,z) \\ &[y(s) - G(s,z)u(s)] \end{aligned} \tag{72}$$

where

$$L(s,z) = P(s,z)X(s,z) + Q(s,z)\tilde{M}(s,z) \tag{73}$$

It can be seen that the observer eqn. 72 consists of two terms. The first term $r(s) = P(s, z)\xi(s) = P(s, z)M^{-1}(s, z)u(s)$ is the estimation for dynamics of the state function $r(t) = E(z)x(t)$ in the disturbance-free case. The second term reflects the mismatch between the measurable output and the disturbance $d(t)$. By properly selecting the gain matrix $L(s, z)$, the estimation error for $r(t)$ can be bounded to a prescribed range.

6 Design example

This example has been worked out in detail in [5] and its references. The disturbance is also added here to illustrate the effects of estimation error. It is desired to obtain the parameterisation of all observers and estimation error dynamics.

Consider the following delay-differential system described by:

$$\dot{x}_1(t) = x_2(t-1) + u(t) + d(t) \tag{74}$$

$$\dot{x}_2(t) = x_1(t-1) + x_2(t) + u(t) \tag{75}$$

$$y(t) = x_2(t) \tag{76}$$

$$r(t) = x_1(t) + x_2(t) \tag{77}$$

Let $zx(t) = x(t-1)$. Then the delay-differential system can be modelled as a linear system over $\mathbf{R}[z]$. The matrices $(C(z), A(z), B(z))$ in state-space expression eqns. 49–51 are given by

$$A(z)=\begin{bmatrix} 0 & z \\ z & 1 \end{bmatrix}, B(z)=\begin{bmatrix} 1 \\ 1 \end{bmatrix}, C(z)=[0 \quad 1] \tag{78}$$

and

$$E(z) = [1 \quad 1], U(z) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{79}$$

Transfer functions are:

$$G(s,z) = \frac{s+z}{s^2 - s - z^2} \quad G_d(s,z) = \frac{z}{s^2 - s - z^2} \tag{80}$$

Let us get the factors of the co-prime factorisation of the system. From eqns. 13 and 14, given $\alpha_i(z) \in \mathbf{R}[z]$, then $F_e(s, z) = [1 - 3z + z^2 \quad -4 + 3z - z^2 \quad 0] \in M(\Theta(z))$ can be obtained such that:

$$\det(sI_e - A_0(s,z)) = \prod_{i=1}^{n}(s + \alpha_i(z)) = s^2 + 2s + 1 \tag{81}$$

And given $\beta_i(z) \in \mathbf{R}[z]$, then

$$K_e = \begin{bmatrix} -1-z \\ -2-z \\ 0 \end{bmatrix} \in M(\mathbf{R}[z])$$

can be obtained such that

$$\det(sI_e - \tilde{A}_0(z)) = \prod_{i=1}^{n}(S + \beta_i(z)) = (s+z)(s+1) \tag{82}$$

Thus, the related factors of the coprime factorisation of the system are, respectively,

$$M(s,z) = \frac{s^2 - s - z^2}{s^2 + 2s + 1} \quad N(s,z) = \frac{s+z}{s^2+2s+1} \tag{83}$$

$$\tilde{M}(s,z) = \frac{s^2 - s - z^2}{(s+1)(s+z)} \quad \tilde{N}(s,z) = \frac{s+z}{(s+1)(s+z)} \tag{84}$$

$$Y(s,z)=\frac{s^2+(z+4)s+4z}{(s+1)(s+z)} \quad X(s,z)=\frac{(7+z^2)s+(1+4z^2)}{(s+1)(s+z)} \tag{85}$$

and

$$P(s,z) = \frac{2s+2z-1}{s^2+2s+1} \tag{86}$$

Thus the set of all observers is given in eqn. 22, where:

$$F(s,z) = \frac{2s+2z-1}{s^2+2s+1}\frac{s^2+(z+4)s+4z}{(s+1)(s+z)} + Q(s,z)\frac{s+z}{(s+1)(s+z)} \tag{87}$$

$$H(s,z) = \frac{2s+2z-1}{s^2+2s+1}\frac{(7+z^2)s+(1+4z^2)}{(s+1)(s+z)} + Q(s,z)\frac{s^2-s-z^2}{(s+1)(s+z)} \tag{88}$$

The parameterisation of estimation error dynamics is given in eqn. 66, where

$$T(s,z) = \frac{s+2z+1}{(s+1)(s+z)} - R(s,z)\frac{z}{(s+1)(s+z)} \tag{89}$$

The plot in Fig. 1 shows unity step responses of estimation error dynamics to disturbance d when three different $R(s, z)$s are chosen. It can be seen that the estimation error dynamics are stable as they are desired despite different $R(s, z)$s. It is apparent that, if a certain performance specification is given, the estimation error can be minimised by properly selecting $R(s, z)$.

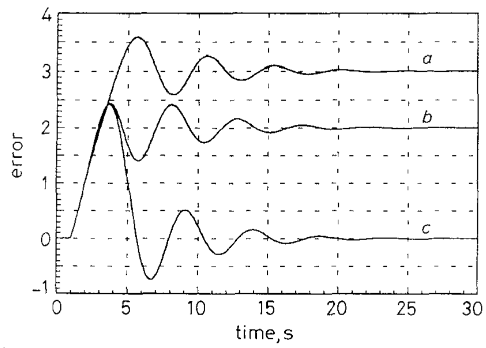

Fig. 1 *Unity step response of estimation error dynamics to disturbance d*
a $R(s, z) = 1/(s+1)(s+z)$
b $R(s, z) = 1/(s+z)$
c $R(s, z) = (s+2z+1)/(s+1)(s+z)$

7 Conclusions

The observer parametrisation of time delay systems is achieved by using the factorisation approach. This provides a dual result to the parametrisation of stabilising controllers of time delay systems. It is also an extension of the observer parameterisation results for the system without time delay in [9, 10].

Although the result of observer parameterisation here can only treat the systems with strictly proper transfer function matrices, it could also be extended to the systems with proper transfer function matrices.

The parameterisation of observer and estimation error for time delay systems obtained in this paper also provides a useful tool for designing an optimal observer in terms of a certain performance specification. It is also suitable for the development of other systematic observer design methods, such as simultaneous state observation for a given set of systems [16].

It should be pointed out that the results presented in this paper are based on the factorisation approach of time delay systems under state–space representation, and therefore, it can be easily realised and implemented with the aid of modern computer aided control system design packages.

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9 Appendix: Proof of Theorem 3

From eqn. 65, we have

$$\begin{aligned} T_1(s,z) &= F_d(s,z) - P(s,z)X(s,z)G_d(s,z) \\ &= E_e(z)\big(sI_e - A_d(z)\big)^{-1}U_e(z) \\ &\quad - E_e(z)\big(sI_e - A_0(s,z)\big)^{-1}B_e(z)F_e(s,z) \\ &\quad \times \big(sI_e - \tilde{A}_0(z)\big)^{-1}K_e(z)C_e(z)\big(sI_e - A_d(z)\big)^{-1}U_e(z) \\ &= E_e(z)\Big[I - \big(sI_e - A_0(s,z)\big)^{-1}B_e(z)F_e(s,z) \\ &\quad \big(sI_e - \tilde{A}_0(z)\big)^{-1}K_e(z)C_e(z)\Big] \\ &\quad \times \big(sI_e - A_d(z)\big)^{-1}U_e(z) \end{aligned} \tag{90}$$

where

$$\begin{aligned} T_{10}(s,z) &= \Big[I - \big(sI_e - A_0(s,z)\big)^{-1} \\ &\quad B_e(z)F_e(s,z)\big(sI_e - \tilde{A}_0(z)\big)^{-1}K_e(z)C_e(z)\Big] \\ &= \Big[I - \big(sI_e - A_e(s,z) - B_e(z)F_e(s,z)\big)^{-1} \\ &\quad B_e(z)F_e(s,z)\big(sI_e - A_e(z) - K_e(z)C_e(z)\big)^{-1} \\ &\quad K_e(z)C_e(z)\Big] \\ &= I - \Big(I - \big(sI_e - A_e(z)\big)^{-1}B_e(z)F_e(s,z)\Big)^{-1} \\ &\quad \times \big(sI_e - A_e(z)\big)^{-1}B_e(z)F_e(s,z) \\ &\quad \times \Big(I - \big(sI_e - A_e(z)\big)^{-1}K_e(z)C_e(z)\Big)^{-1} \\ &\quad \times \big(sI_e - A_e(z)\big)^{-1}K_e(z)C_e(z) \end{aligned} \tag{91}$$

Let us denote:

$$a = \big(sI_e - A_e(z)\big)^{-1}B_e(z)F_e(s,z) \tag{92}$$

$$b = \big(sI_e - A_e(z)\big)^{-1}K_e(z)C_e(z) \tag{93}$$

Then

$$\begin{aligned} T_{10}(s,z) &= I - (I-a)^{-1}a(I-b)^{-1}b \\ &= (I-a)^{-1}\big(I - a - a(I-b)^{-1}b\big) \\ &= (I-a)^{-1}\big(I - a(I + (I-b)^{-1}b)\big) \\ &= (I-a)^{-1}\big(I - a(I-b)^{-1}\big) \\ &= (I-a)^{-1}(I-b-a)(I-b)^{-1} \end{aligned} \tag{94}$$

Substituting eqns. 92 and 93 into eqn. 94, gives

$$\begin{aligned} T_{10}(s,z) &= \Big(I - \big(sI_e - A_e(z)\big)^{-1}B_e(z)F_e(s,z)\Big)^{-1} \\ &\quad \times \Big(I - \big(sI_e - A_e(z)\big)^{-1}K_e(z)C_e(z) \\ &\quad - \big(sI_e - A_e(z)\big)^{-1}B_e(z)F_e(s,z)\Big) \\ &\quad \times \Big(I - \big(sI_e - A_e(z)\big)^{-1}K_e(z)C_e(z)\Big)^{-1} \\ &= \big(sI_e - A_e(z) - B_e(z)F_e(s,z)\big)^{-1}\big(sI_e - A_e(z)\big) \\ &\quad \times \big(sI_e - A_e(z)\big)^{-1}\big(sI_e - A_e(z) \\ &\quad - B_e(z)F_e(s,z) - K_e(z)C_e(z)\big) \\ &\quad \times \big(sI_e - A_e(z) - K_e(z)C_e(z)\big)^{-1}\big(sI_e - A_e(z)\big) \\ &= \big(sI_e - A_e(z) - B_e(z)F_e(s,z)\big)^{-1} \\ &\quad \times \big(sI_e - A_e(z) - B_e(z)F_e(s,z) - K_e(z)C_e(z)\big) \\ &\quad \times \big(sI_e - A_e(z) - K_e(z)C_e(z)\big)^{-1}\big(sI_e - A_e(z)\big) \end{aligned} \tag{95}$$

Substituting eqn. 95 into eqn. 90, we have

$$\begin{aligned} T_1(s,z) &= E_e(z)T_{10}(s,z)\big(sI_e - A_d(z)\big)^{-1}U_e(z) \\ &= E_e(z)\big(sI_e - A_e(z) - B_e(z)F_e(s,z)\big)^{-1} \\ &\quad \times \big(sI_e - A_e(z) - B_e(z)F_e(s,z) - K_e(z)C_e(z)\big) \\ &\quad \times \big(sI_e - A_e(z) - K_e(z)C_e(z)\big)^{-1}\big(sI_e - A_e(z)\big) \\ &\quad \times \big(sI_e - A_d(z)\big)^{-1}U_e(z) \end{aligned} \tag{96}$$

Since

$$\begin{aligned} &\big(sI_e - A_e(z)\big)\big(sI_e - A_d(z)\big)^{-1} \\ &= \begin{pmatrix} sI - A(z) & -B(z) \\ 0 & I \end{pmatrix}\begin{pmatrix} sI - A(z) & -U(z) \\ 0 & I \end{pmatrix}^{-1} \\ &= \begin{pmatrix} sI - A(z) & -B(z) \\ 0 & I \end{pmatrix} \\ &\times \begin{pmatrix} (sI - A(z))^{-1} & (sI - A(z))^{-1}U(z) \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} I & U(z) - B(z) \\ 0 & I \end{pmatrix} \end{aligned} \tag{97}$$

Then eqn. 96 becomes

$$\begin{aligned} T_1(s,z) &= E_e(z)\big(sI_e - A_e(z) - B_e(z)F_e(s,z)\big)^{-1} \\ &\quad \times \big(sI_e - A_e(z) - B_e(z)F_e(s,z) - K_e(z)C_e(z)\big) \\ &\quad \times \big(sI_e - A_e(z) - K_e(z)C_e(z)\big)^{-1} \\ &\quad \times \begin{pmatrix} I & U(z) - B(z) \\ 0 & I \end{pmatrix}U_e(z) \end{aligned} \tag{98}$$

From eqn. 97, $T_2(s, z)$ in eqn. 65 can be rewritten as

$$
\begin{aligned}
T_2(s,z) &= \tilde{M}(s,z)G_d(s,z) \\
&= \tilde{M}(s,z)C_e(z)\big(sI_e - A_d(z)\big)^{-1}U_e(z) \\
&= \tilde{M}(s,z)C_e(z)\big(sI_e - A_e(z)\big)^{-1} \\
&\quad \times \begin{pmatrix} I & U(z)-B(z) \\ 0 & I \end{pmatrix} U_e(z)
\end{aligned} \tag{99}
$$

From eqns. 15 and 24, we have

$$
\begin{aligned}
&C_e(z)(sI_e - A_e(z))^{-1}B_e(z) = \tilde{M}^{-1}(s,z)\tilde{N}(s,z) \\
&= \tilde{M}^{-1}(s,z)C_e(z)\left(sI_e - \tilde{A}_0(s,z)\right)^{-1}B_e(z)
\end{aligned}
$$

Then

$$
\begin{aligned}
&C_e(z)(sI_e - A_e(z))^{-1} \\
&= \tilde{M}^{-1}(s,z)C_e(z)\left(sI_e - \tilde{A}_0(s,z)\right)^{-1}
\end{aligned} \tag{100}
$$

Substituting eqn. 100 into eqn. 99, gives

$$
\begin{aligned}
T_2(s,z) &= \tilde{M}(s,z)\tilde{M}^{-1}(s,z)C_e(z)\left(sI_e - \tilde{A}_0(s,z)\right)^{-1} \\
&\quad \times \begin{pmatrix} I & U(z)-B(z) \\ 0 & I \end{pmatrix} U_e(z) \\
&= C_e(z)\left(sI_e - \tilde{A}_0(s,z)\right)^{-1}\begin{pmatrix} I & U(z)-B(z) \\ 0 & I \end{pmatrix} U_e(z)
\end{aligned} \tag{101}
$$

Thus substituting eqns. 98 and 101 into eqn. 64, we obtain

$$
\begin{aligned}
T(s,z) &= T_1(s,z) - Q(s,z)T_2(s,z) \\
&= E_e(z)\big(sI_e - A_e(z) - B_e(z)F_e(s,z)\big)^{-1} \\
&\quad \times \big(sI_e - A_e(z) - B_e(z)F_e(s,z) - K_e(z)C_e(z)\big) \\
&\quad \times \big(sI_e - A_e(z) - K_e(z)C_e(z)\big)^{-1} \\
&\quad \times \begin{pmatrix} I & U(z)-B(z) \\ 0 & I \end{pmatrix} U_e(z) - Q(s,z)C_e(z) \\
&\quad \times \left(sI_e - \tilde{A}_0(s,z)\right)^{-1}\begin{pmatrix} I & U(z)-B(z) \\ 0 & I \end{pmatrix} U_e(z) \\
&= E_e(z)\left(I - \big(sI_e - A_e(z) - B_e(z)F_e(s,z)\big)^{-1}K_e(z)C_e(z)\right) \\
&\quad \times \big(sI_e - A_e(z) - K_e(z)C_e(z)\big)^{-1}\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix} \\
&\quad - Q(s,z)C_e(z)\left(sI_e - \tilde{A}_0(s,z)\right)^{-1}\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix} \\
&= E_e(z)\left(I - \big(sI_e - A_0(s,z)\big)^{-1}K_e(z)C_e(z)\right) \\
&\quad \times \left(sI_e - \tilde{A}_0(z)\right)^{-1}\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix} \\
&\quad - Q(s,z)C_e(z)\left(sI_e - \tilde{A}_0(s,z)\right)^{-1}\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix} \\
&= E_e(z)\left(\left(sI_e - \tilde{A}_0(z)\right)^{-1} - \big(sI_e - A_0(s,z)\big)K_e(z)C_e(z)\right. \\
&\quad \left. \times \left(sI_e - \tilde{A}_0(z)\right)^{-1}\right)\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix} \\
&\quad - Q(s,z)C_e(z)\left(sI_e - \tilde{A}_0(s,z)\right)^{-1}\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix} \\
&= E_e(z)\left(sI_e - \tilde{A}_0(z)\right)^{-1}\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix} \\
&\quad - E_e(z)\big(sI_e - A_0(s,z)\big)^{-1}K_e(z)C_e(z) \\
&\quad \times \left(sI_e - \tilde{A}_0(z)\right)^{-1}\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix} \\
&\quad - Q(s,z)C_e(z)\left(sI_e - \tilde{A}_0(s,z)\right)^{-1}\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix} \\
&= E_e(z)\left(sI_e - \tilde{A}_0(z)\right)^{-1}\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix} \\
&\quad - \left[E_e(z)\big(sI_e - A_0(s,z)\big)^{-1}K_e(z) + Q(s,z)\right] \\
&\quad \times C_e(z)\left(sI_e - \tilde{A}_0(s,z)\right)^{-1}\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix} \\
&= E_e(z)\left(sI_e - \tilde{A}_0(z)\right)^{-1}\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix} \\
&\quad - R(s,z)C_e(z)\left(sI_e - \tilde{A}_0(s,z)\right)^{-1}\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix}
\end{aligned} \tag{102}
$$

where

$$
R(s,z) = E_e(z)\big(sI_e - A_0(s,z)\big)^{-1}K_e(z) + Q(s,z) \tag{103}
$$

In eqn. 102, let

$$
T_{11}(s,z) = E_e(z)\left(sI_e - \tilde{A}_0(z)\right)^{-1}\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix} \tag{104}
$$

$$
T_{22}(s,z) = C_e(z)\left(sI_e - \tilde{A}_0(s,z)\right)^{-1}\begin{pmatrix} U(z)-B(z) \\ I \end{pmatrix} \tag{105}
$$

They are the matrices in $\mathbf{S}_{IN}$. Thus eqn. 66 in Theorem 3 is obtained. This completes the proof.